

Approximation to Distribution of Product of Random Variables Using Orthogonal Polynomials for Lognormal Density

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Abstract—We derive a closed-form expression for the orthogonal polynomials associated with the general lognormal density. The result can be utilized to construct easily computable approximations for probability density function of a product of random variables, when the considered variates are either independent or correlated. As an example, we have calculated the approximative distribution for the product of Nakagami- m variables. Simulations indicate that accuracy of the proposed approximation is good with small cross-correlations under light fading condition.

Index Terms—Product of random variables, central limit theorem, lognormal distribution, orthogonal polynomials, Nakagami- m .

I. INTRODUCTION

STATISTICAL properties of products of Random Variables (RVs) are essential in performance analysis of contemporary wireless communication systems. For example, the fading amplitude of multi-hop relaying systems follows the distribution of the product of Nakagami- m RVs [1]. In addition, the cascaded-keyhole channel can be modeled using the product of individual keyhole channels [2].

The exact Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the product of independent Beta, Gamma and Gaussian RVs can be represented in terms of the Meijer-G function [3]. A more general framework involving the Fox H-function was proposed in [4] for the distribution of product of almost any non-negative independent RVs. Although being theoretically interesting, these G- or H-function representations are difficult to evaluate on a low-complexity computing platform, where the calculation of the product statistics is required [5]. The numerical calculations of G- and H-functions require a numerical solution of contour integrals or a construction of look-up tables that cover all parameter combinations. Both approaches set stringent requirements for the considered platform either in the computing capability or the memory capacity. Recently, Ahmed *et al.* [6] employed the Mellin transform and the residue theorem to derive the PDF of a product of *i.i.d.* Nakagami- m RVs as an infinite series. By exploiting the structure of the Mellin transforms, the authors in [7], [8] proposed approximations to the PDFs and CDFs of products of independent Rayleigh,

Gamma, Nakagami- m and Gaussian RVs. However, when the RVs in product are correlated, exact distributions are known only in some special cases (e.g. bivariate Nakagami- m in [9]). For arbitrary number of RVs with generic correlations, none of the aforementioned methodologies are suitable to give a tractable solution. This is the main issue to be addressed in this paper.

Motivated by the Central Limit Theorem (CLT), the approximation for the product of RVs can be constructed by using a lognormal density and its associated orthogonal polynomials. We apply the approach of [10] where the first few moments of the product and the approximated density function are set equal. The resulting approximative distribution involves only a finite sum of polynomials and the elementary lognormal density function. Therefore, implementing the proposed expressions on practical computing platforms is straightforward compared with the G- or H-function representations. We note that the proposed framework is suitable for cases of both independent and correlated variables provided that moments of a product can be computed in closed-form.

In this paper we deduce a closed-form expression for the orthogonal polynomials associated with a general lognormal density. The result is obtained by taking advantage of the determinant representation of orthogonal polynomials. To give a specific example, we derive an approximative PDF and CDF for the product of both independent and correlated Nakagami- m RVs. Numerical results are compared with simulations in terms of the Complementary CDF (CCDF). The overall approximation accuracy is measured by the Mean Square Error (MSE) between the approximative CDF and the empirical CDF. The orthogonal polynomials associated with a standard lognormal density¹ were first developed in [11], which is a special case of our result. It is worth of noticing that our results are not straightforward extensions of the results presented in [11] but require new methodological elements.

II. ORTHOGONAL POLYNOMIALS ASSOCIATED WITH GENERAL LOGNORMAL DENSITY

Assume that Y is a Gaussian RV with mean μ and variance σ^2 . Then $X = e^Y$ follows the lognormal distribution

$$f_{\text{LN}}(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}, \quad x \in (0, \infty) \quad (1)$$

¹When the corresponding Gaussian distribution is zero mean and unit variance.

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and the i -th moment, ν_i ($i \in \mathbb{N}$), of X is given by

$$\nu_i = \int_0^\infty x^i f_{\text{LN}}(x) dx = e^{i\mu + \frac{1}{2}i^2\sigma^2}. \quad (2)$$

Let $\pi_n(x)$ be the n -th degree polynomial

$$\pi_n(x) = \sum_{k=0}^n c_{n,k} x^k, \quad (3)$$

where $c_{n,n} \neq 0$. The polynomials $\{\pi_n(x)\}$ are said to be orthogonal with respect to the general lognormal density if

$$\int_0^\infty \pi_j(x) \pi_k(x) f_{\text{LN}}(x) dx = h_j \delta_{jk}, \quad j, k \in \mathbb{N}, \quad (4)$$

where $h_j = \sum_{i=0}^j \sum_{k=0}^j c_{j,i} c_{j,k} \nu_{i+k}$ is a normalization factor. The symbol δ_{jk} is the Kronecker delta symbol, which is defined as $\delta_{jk} = 1$ if $j = k$ and zero otherwise.

Due to (2) any arbitrary moment of $f_{\text{LN}}(x)$ exists. Thus, functions x^i ($i \in \mathbb{N}$) belong to the space of square integrable functions with respect to the weight function $f_{\text{LN}}(x)$. There exists a unique set of orthogonal polynomials $\{\pi_n(x)\}$ which admits the explicit determinant representation $\pi_n(x) = (\Delta_n)^{-1} \Delta_n(x)$ [12, Th. 2.1.1], where $\Delta_n(x)$ denotes the n -th degree polynomial

$$\Delta_n(x) = \begin{vmatrix} \nu_0 & \cdots & \nu_{n-1} & \nu_n \\ \vdots & \ddots & \vdots & \vdots \\ \nu_{n-1} & \cdots & \nu_{2n-2} & \nu_{2n-1} \\ 1 & \cdots & x^{n-1} & x^n \end{vmatrix}, \quad (5)$$

and the constant Δ_n is formed by deleting the last row and column from $\Delta_n(x)$, i.e. $\Delta_n = |\nu_{i+j}|_{i,j=0,\dots,n-1}$ with $\Delta_0 = 1$. The determinant (5) can be expanded using the cofactors with respect to the last row. Then $\pi_n(x)$ becomes

$$\pi_n(x) = \sum_{k=0}^n (-1)^{n+k} \frac{\Delta_{n,k}}{\Delta_n} x^k, \quad (6)$$

where the cofactor $\Delta_{n,k}$ with respect to x^k is obtained by removing the $(k+1)$ -th column and the last row from (5), $\Delta_{0,0} = 0$ and $\Delta_{1,0} = 1$. After comparing (6) with (3), we obtain

$$c_{n,k} = (-1)^{n+k} \frac{\Delta_{n,k}}{\Delta_n}, \quad k = 0, \dots, n. \quad (7)$$

Let us calculate an explicit expression for $c_{n,k}$. First, by the definition of $\Delta_{n,k}$ and Δ_n it is observed that $c_{n,n} = 1$, i.e. the orthogonal polynomials (6) are monic². Furthermore, after inserting (2) into $\Delta_{n,k}$ and denoting $q = e^{\sigma^2}$, we obtain (see Appendix for details)

$$\Delta_{n,k} = E(n) \frac{\nu_n \prod_{i=0}^{n-1} \prod_{j=i+1}^n (q^j - q^i)}{\nu_k \prod_{j=k+1}^n (q^j - q^k) \prod_{i=0}^{k-1} (q^k - q^i)}, \quad (8)$$

where $E(n) = e^{n(n-1)\mu + \frac{\sigma^2}{6}n(2n^2-3n+1)}$. For $k = n$, $\Delta_n = \Delta_{n,n}$ and (8) becomes

$$\Delta_n = E(n) \prod_{i=0}^{n-2} \prod_{j=i+1}^{n-1} (q^j - q^i). \quad (9)$$

²Monic polynomial is defined as the polynomial where the coefficient of highest degree term is unity.

After substituting (8) and (9) into (7) we find that

$$c_{n,k} = (-1)^{n+k} e^{(n-k)\mu} q^{(n-1/2)(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad (10)$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$ is the generalized binomial coefficient.

III. APPROXIMATION TO THE DENSITY OF PRODUCT OF NAKAGAMI- m RANDOM VARIABLES

A. General Framework

According to the CLT, the distribution of a product of RVs can be approximated by a lognormal distribution when the number of RVs is large. Motivated by this, we choose $f_{\text{LN}}(x)$ as an initial approximation for the distribution of a product of RVs in the context of the moment based density approximation, which is derived in [10]. Let $M(k)$ denote the k -th moment of a density function $f(x)$, the approximation to $f(x)$ reads

$$f(x) \simeq f_{\text{LN}}(x) \sum_{i=0}^N \eta_i \pi_i(x), \quad (11)$$

where the constant $\eta_i = \frac{1}{h_i} \sum_{k=0}^i c_{i,k} M(k)$. Note that the first N moments of the approximation (11) are matched with the corresponding moments of $f(x)$ [10].

Alternatively, (11) can be rearranged by combining the coefficients of $\pi_i(x)$ with the same power of x , resulting in $f(x) \simeq f_{\text{LN}}(x) \sum_{i=0}^N \xi_i x^i$ where $\xi_j = \sum_{k=j}^N c_{k,j} \eta_k$. After direct integration the approximated CDF attains the form

$$F(x) \simeq \sum_{i=0}^N \xi_i \nu_i \Phi\left(\frac{\log(x) - \mu}{\sigma} - i\sigma\right), \quad (12)$$

where $\Phi(\cdot)$ is the CDF of a standard normal RV. The CCDF $\bar{F}(x) = 1 - F(x)$ is approximated by replacing $F(x)$ with the expression (12).

It is a typical situation that the moments of certain RV are relatively easy to obtain whereas its exact distribution is difficult to calculate or unavailable [13]. The proposed approximations (11) and (12) are particularly useful in this situation, as will be shown in the next subsection.

B. Moment Based Approximation to Distribution of Product of Nakagami- m RVs

Let RV $P = \prod_{i=1}^K R_i$ be a product of K Nakagami- m RVs each with the PDF

$$f_{R_i}(x) = \frac{2m_i^{m_i} x^{2m_i-1}}{\Omega_i^{m_i} \Gamma(m_i)} \exp\left(-\frac{m_i}{\Omega_i} x^2\right), \quad x \in (0, \infty). \quad (13)$$

We first consider the case when the variables R_i are correlated with each other. In the literature, there exists different representations for the joint PDF of multivariate Nakagami- m RVs, which are either limited in parameter values or cross-correlation structures. In this paper, we adopt a recent result derived in [14], which gives the joint PDF as a single integral. In the proposed approach, the RVs R_i are assumed to have the same fading parameter m and the joint PDF is valid

for integer and half-integer values of m . The power cross-correlation coefficient between R_i^2 and R_j^2 is of the form

$$\rho_{R_i^2, R_j^2} = \frac{\mathbb{E}[R_i^2 R_j^2] - \mathbb{E}[R_i^2] \mathbb{E}[R_j^2]}{\sqrt{\text{Var}[R_i^2] \text{Var}[R_j^2]}} = \lambda_i^2 \lambda_j^2. \quad (14)$$

Note that the derived approximation framework is not limited by the above multivariate model.

Based on the joint PDF [14, eq. (20)], the k -th moment of the RV P is calculated by using [15, eq. (6.643/2)] as

$$M(k) = \frac{\Gamma(m + k/2)^K}{m^{Kk/2} \Gamma(m)^{K+1}} \prod_{i=1}^K [\Omega_i (1 - \lambda_i^2)]^{\frac{k}{2}} \times \int_0^\infty t^{m-1} e^{-t} \prod_{i=1}^K {}_1F_1\left(-\frac{k}{2}, m, \frac{\lambda_i^2 t}{\lambda_i^2 - 1}\right) dt, \quad (15)$$

where ${}_1F_1(a, b, c)$ denotes the Kummer confluent hypergeometric function. The parameters μ and σ^2 of $f_{\text{LN}}(x)$ can be obtained by equating them with the mean and variance of $\log(P)$ respectively as

$$\mu = \sum_{i=1}^K \mathbb{E}[\log(R_i)] = \frac{1}{2} \sum_{i=1}^K \left[\Psi_0(m) - \log\left(\frac{m}{\Omega_i}\right) \right], \quad (16)$$

$$\sigma^2 = \sum_{i=1}^K \text{Var}[\log(R_i)] + 2 \sum_{i < j} \text{Cov}[\log(R_i), \log(R_j)] = \frac{1}{4} \sum_{i=1}^K \Psi_1(m) + 2 \sum_{i < j} \left(\int_0^\infty \frac{I_i(t) I_j(t) t^{m-1}}{4 \Gamma(m) e^t} dt - \zeta_i \zeta_j \right), \quad (17)$$

where $\Psi_n(x)$ is the polygamma function that is defined as the $(n+1)$ -th derivative of the logarithm of $\Gamma(x)$, and $\zeta_i = \frac{1}{2} \left[\Psi_0(m) - \log\left(\frac{m}{\Omega_i}\right) \right]$. Here the function $I_i(t)$ is

$$I_i(t) = \Psi_0(m) + \log\left(\frac{\Omega_i(1 - \lambda_i^2)}{m}\right) - {}_1F_1'\left(a, m, \frac{\lambda_i^2 t}{\lambda_i^2 - 1}\right) \Big|_{a=0},$$

where ${}_1F_1'(a, b, c)$ refers to the derivative of ${}_1F_1(a, b, c)$ with respect to the parameter a .

Next consider the case where the RVs R_i are independent. We notice that with $\lambda_i = 0$, the joint PDF [14, eq. (20)] is in the form of a product of individual PDFs of R_i ($i = 1, 2, \dots, K$), which implies statistical independence. Thus, we let $\lambda_i = 0$ and calculate the moments of P from (15)

$$M(k) = \prod_{i=1}^K \frac{\Gamma(m + k/2)}{\Gamma(m)} \left(\frac{\Omega_i}{m}\right)^{k/2}. \quad (18)$$

While the calculation of μ is not affected by the independency and it is given by expression (16), the variance σ^2 is reduced to the form

$$\sigma^2 = \frac{1}{4} \sum_{i=1}^K \Psi_1(m) \quad (19)$$

with all the covariances $\text{Cov}[\log(R_i), \log(R_j)]$ in (17) equal to zero. It is noted that (18) and (19) agree with the results given by [1].

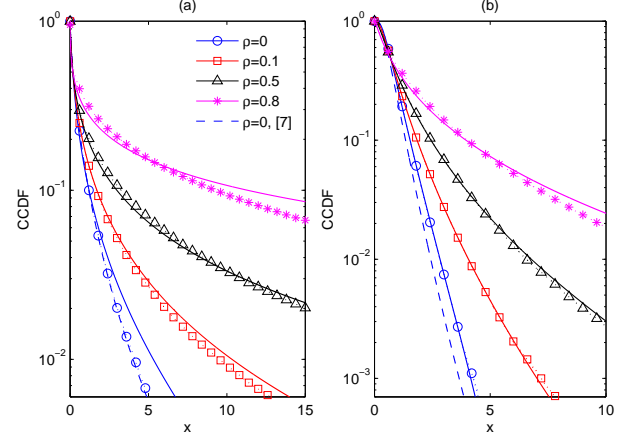


Fig. 1. CCDF of products of six Nakagami- m RVs when $\Omega = 1$ and $N = 16$. (a) $m = 1$; (b) $m = 4$. Solid lines: approximative PDFs; markers: simulated PDFs.

IV. NUMERICAL RESULTS

In Fig. 1, we compare simulations with the approximative CCDFs for the product of six Nakagami- m RVs with parameters $m = 1, 4$ and $\Omega = 1$, using orthogonal polynomials of up to 16-th degree. Here we consider equal cross-correlations, i.e. $\rho_{R_i^2, R_j^2} = \rho$ ($i \neq j$), and calculate the parameters of the approximation using (15), (16) and (17) with $\rho = 0.1, 0.5$ and 0.8 for correlated Nakagami- m RVs. In case of independent RVs (with $\rho = 0$), equations (18), (16) and (19) are applied. When $m = 1$ the RVs R_i are Rayleigh distributed, while with $m = 4$ the R_i are approximately Rician distributed with the Rician κ -factor given by $\kappa = 6.46$ [5]. For each simulated CCDF curve, we generate 10^6 realizations of the Nakagami- m RVs R_i ($i = 1, \dots, 6$) using Sim's method [16]. As a comparison, we also plot the approximative CCDFs calculated from [8, eq. (36)] for the cases with independent Nakagami- m RVs. Fig. 1(b) shows that differences between the approximative CCDFs and simulations are less than 10^{-3} when $\rho = 0$ and 0.1 , and less than 10^{-2} when $\rho = 0.5$. In addition, when $m = 4$ and $\rho = 0$, the proposed approximation yields improved accuracy compared with the results given in [8]. However, as $m = 1$, the approximations with the same correlation coefficients are noticeably inaccurate due to slow convergence of the series in (12). Note that in both cases the approximative CCDFs with $\rho = 0.8$ deviate from the simulations since the CLT fails under high correlation condition.

Table I summarizes the MSE $\epsilon^2 = \int_0^\infty |F^*(x) - F(x)|^2 dF(x)$ between the approximation $F(x)$ and the empirical distribution function $F^*(x)$ of the product P as a function of the number of RVs K . When K is small ($K \leq 10$), Table I shows that the approximated CDF achieves considerably smaller MSE in low-correlation cases ($\rho = 0.1$ and 0.5) compared with the independent case ($\rho = 0$). As K increases, the accuracy for the independent case becomes superior to those of correlated ones. When $m = 4$, all approximations are much more accurate compared with the corresponding Rayleigh cases ($m = 1$), and the MSEs ϵ^2 , except for those

TABLE I
MEAN SQUARE ERROR ϵ^2 OF APPROXIMATIVE DISTRIBUTION

K	2	4	6	8	10	12	14	16	18	20
$m = 1, \rho = 0$	1.14e-3	1.09e-3	6.28e-4	3.78e-4	2.72e-4	2.02e-4	1.67e-4	1.45e-4	1.28e-4	1.14e-4
$m = 1, \rho = 0.1$	1.06e-3	5.00e-4	1.05e-4	1.66e-5	1.73e-5	5.35e-5	1.03e-4	1.72e-4	2.40e-4	3.21e-4
$m = 1, \rho = 0.5$	9.99e-4	1.68e-4	2.10e-5	9.54e-5	2.07e-4	3.18e-4	4.16e-4	5.36e-4	6.24e-4	6.98e-4
$m = 1, \rho = 0.8$	2.17e-3	9.16e-4	7.72e-4	7.41e-4	7.28e-4	7.21e-4	7.09e-4	7.15e-4	7.13e-4	7.16e-4
$m = 4, \rho = 0$	8.13e-6	2.29e-5	2.43e-5	1.64e-5	2.24e-5	3.15e-5	4.34e-5	4.78e-5	5.14e-5	5.02e-5
$m = 4, \rho = 0.1$	8.31e-6	1.16e-5	2.65e-6	2.33e-6	1.77e-5	4.95e-5	8.93e-5	1.15e-4	1.31e-4	1.47e-4
$m = 4, \rho = 0.5$	7.03e-6	1.34e-5	7.44e-6	9.73e-6	1.74e-5	2.48e-5	3.36e-5	4.21e-5	4.99e-5	5.49e-5
$m = 4, \rho = 0.8$	3.58e-5	3.20e-4	4.63e-4	3.03e-4	2.27e-4	2.12e-4	2.09e-4	2.11e-4	2.09e-4	2.06e-4

where $K > 16$ and $\rho = 0.1$, is less than 10^{-4} in case of independent RVs or when correlation is low.

Although the accuracies of the approximations (11) and (12) should be improved as N increases, analysis of the exact improvement is difficult due to the multiple nested finite summations involved. However, numerical results indicate that the i -th term in the summation (12) converges quickly to zero since the coefficient $\xi_i \nu_i$ converges to zero and the CDF $\Phi\left(\frac{\log(x)-\mu}{\sigma} - i\sigma\right)$ is bounded. The value $N = 16$ is shown to be sufficient to yield a stable CDF approximation.

V. CONCLUSION

Knowledge of the distributions of product of random variables is important for understanding the performance of various communication systems. In this work, we first derived a closed-form expression for the orthogonal polynomials associated with general lognormal density. The derived result was subsequently applied in approximating the distribution of the product of random variables. As an example, we calculated closed-form approximations for the distributions of product of Nakagami- m variates. Under light fading conditions, the resulting expressions achieve a good trade-off between computation complexity and approximation accuracy for small cross-correlations. It is still an open problem to find a closed-form (approximated) distribution for the product of correlated Nakagami- m RVs under deep fading condition.

APPENDIX

We note that the determinant

$$\Delta_{n,k} = \left| e^{(i+j)\mu + \frac{(i+j)^2}{2}\sigma^2} \right|_{i=0,\dots,n-1; j=0,\dots,n; j \neq k} \quad (20)$$

remains unchanged by factoring out the term $e^{i\mu + i^2\sigma^2/2}$ from the i -th row and the term $e^{j\mu + j^2\sigma^2/2}$ from the j -th column. Namely, we have

$$\Delta_{n,k} = E(n) \frac{\nu_n}{\nu_k} \left| e^{ij\sigma^2} \right|_{i=0,\dots,n-1; j=0,\dots,n; j \neq k} \quad (21)$$

$$= E(n) \frac{\nu_n \prod_{i=0}^{n-1} \prod_{j=i+1}^n (q^j - q^i)}{\nu_k \prod_{j=k+1}^n (q^j - q^k) \prod_{i=0}^{k-1} (q^k - q^i)}, \quad (22)$$

where $E(n) = e^{n(n-1)\mu + \frac{\sigma^2}{6}n(2n^2-3n+1)}$ and the second equality follows from the fact that $|e^{ij\sigma^2}|$ is a Vandermonde determinant.

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REFERENCES

- [1] G. K. Karagiannidis, N. C. Sagias and P. T. Mathiopoulos, "N*Nakagami: A novel stochastic model for cascaded fading channels," *IEEE Trans. Commun.*, vol. 55, no. 8, pp. 1453-1458, Aug. 2007.
- [2] D. Chizhik, G. J. Foschini, M. J. Gans and R. A. Valenzuela, "Key-holes, correlations, and capacities of multielement transmit and receive antennas," *IEEE Trans. Wireless Commun.*, vol. 1, no. 2, pp. 361-368, Apr. 2002.
- [3] M. D. Springer and W. E. Thompson, "The distribution of products of Beta, Gamma and Gaussian random variables," *SIAM J. Applied Maths.*, vol. 18, no. 4, pp. 721-737, Jun. 1970.
- [4] B. D. Carter and M. D. Springer, "The distribution of products, quotients and powers of independent H-function variates," *SIAM J. Applied Maths.*, vol. 33, no. 4, pp. 542-558, Dec. 1977.
- [5] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*, 2nd ed. New York: Wiley, 2004.
- [6] S. Ahmed, L.-L. Yang and L. Hanzo, "Probability distributions of products of Rayleigh and Nakagami- m variables using Mellin transforms," in *Proc. 2011 IEEE Int. Conf. Commun.*
- [7] H. Lu, Y. Chen and N. Cao, "Accurate approximation to the PDF of the product of independent Rayleigh random variables," *IEEE Antennas Wireless Propag. Lett.*, vol. 10, pp. 1019-1022, Oct. 2011.
- [8] Y. Chen, G. K. Karagiannidis, H. Lu and N. Cao, "Novel approximations to the statistics of products of independent random variables and their applications in wireless communications," *IEEE Trans. Veh. Technol.*, vol. 61, no. 2, pp. 443-454, Feb. 2012.
- [9] M. Nakagami and M. Ōta, "The distribution of the product of two correlated m -variables," *Radio Wave Propagation Res. Committee Japan Rep.*, vol. 36, pp. 304-320, 1957.
- [10] S. Provost and H. T. Ha, "On the inversion of certain moment matrices," *Linear Algebra and its Applications*, vol. 430, no. 10, pp. 2650-2658, May 2009.
- [11] O. G. Ernst, A. Mugler, H. Starkloff and E. Ullmann, "On the convergence of generalized polynomial chaos expansions," *ESAIM Mathematical Modelling and Numerical Analysis*, vol. 46, no. 2, pp. 317-339, Oct. 2011.
- [12] G. Szegő, *Orthogonal Polynomials*. New York: Amer. Math. Soc., 1939.
- [13] H. T. Ha, *Advances in Moment-Based Density Approximation Methods*. Ph.D thesis, University of Western Ontario, 2006.
- [14] N. C. Beaulieu and K. T. Hemachandra, "Novel simple representations for Gaussian class multivariate distributions with generalized correlation," *IEEE Trans. Inf. Theory*, vol. 57, no. 12, pp. 8072-8083, Dec. 2011.
- [15] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. New York: Academic, 2007.
- [16] C. H. Sim, "Generation of Poisson and Gamma random vectors with given marginal and covariance matrix," *J. Stat. Computat. Simulat.*, vol. 47, no. 1, pp. 1-10, 1993.